

# Zero-sum linear quadratic stochastic integral games and BSVIEs \*

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May 28 2010

## Abstract

This paper formulates and studies a linear quadratic (LQ for short) game problem governed by linear stochastic Volterra integral equation. Sufficient and necessary condition of the existence of saddle points for this problem are derived. As a consequence we solve the problems left by Chen and Yong in [3]. Firstly, in our framework, the term  $GX^2(T)$  is allowed to be appear in the cost functional and the coefficients are allowed to be random. Secondly we study the unique solvability for certain coupled forward-backward stochastic Volterra integral equations (FBSVIEs for short) involved in this game problem. To characterize the condition aforementioned explicitly, some other useful tools, such as backward stochastic Fredholm-Volterra integral equations (BSFVIEs for short) and stochastic Fredholm integral equations (FSVIEs for short) are introduced. Some relations between them are investigated. As a application, a linear quadratic stochastic differential game with finite delay in the state variable and control variables is studied.

*Keywords:* Stochastic integral games, open-loop controls, saddle points, linear quadratic optimal control problem, coupled forward-backward stochastic Volterra integral equations, backward stochastic Fredholm-Volterra integral equation

## 1 Introduction

Differential game is a classical problem, there are several frameworks of investigating it as far as the strategies are concerned; see [4], [7], [8], [10], [12] for open-loop strategies of both deterministic and stochastic differential games, and [2], [5], [8], [10], [12], [21], [22] for closed-loop strategy counterpart. In addition, we would also like to mention the work of Fleming and Souganidis [6], who firstly gave a study on two-player zero-sum stochastic differential games. Nonetheless there are few literature to demonstrate some

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\*This work is supported by National Natural Science Foundation of China Grant 10771122, Natural Science Foundation of Shandong Province of China Grant Y2006A08 and National Basic Research Program of China (973 Program, No. 2007CB814900).

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analytical research on the more general dynamic setting, such as the system driven by a Volterra equation. In this connection the only paper we know is You [29], where the problem is spread out in the deterministic setting. Such a lack of study is certainly not due to the unimportance and non-interestingness of the problem, rather it is because, we believe, that most of the effective techniques in the conventional differential game are not analyzable and applicable well in such setting. For example, as compared with differential dynamic system, the time-consistency (or semi-group) property is failure in the Volterra integral case, thus many good results along this no longer hold.

In this paper, we will initiate a study on zero-sum linear quadratic (LQ for short) stochastic integral game. A particularly important notion for investigating the problem is given by Nash equilibria. Loosely speaking, in the game problem, Player 1 wishes to minimize the quadratic performance (4) which represents the cost and Player 2 wishes to maximize (4) which represents the payoff. Since both of the two players are non-cooperative, they would like to seek their admissible controls  $\hat{u}_1$  and  $\hat{u}_2$ , respectively, such that

$$J(\hat{u}_1, u_2) \leq J(\hat{u}_1, \hat{u}_2) \leq J(u_1, \hat{u}_2), \quad (1)$$

for all the admissible controls  $u_1$  and  $u_2$ . The reason for (1) holds is that none of the players can improve his/her outcome  $J(\hat{u}_1, \hat{u}_2)$  by deviating from  $\hat{u}_1$  or  $\hat{u}_2$  unilaterally. Thus both players will be satisfied with the controls  $\hat{u}_1$  and  $\hat{u}_2$ , respectively. In our framework, we refer to  $(\hat{u}_1, \hat{u}_2)$  as an open-loop saddle point of the game over  $[0, T]$ , and we consider only the open loop strategies in the following part. Additionally, we point out that in general, an open-loop saddle point (if it exists) is not necessarily unique.

Before demonstrating the model we will study in this paper, we would like to summarize how our work relates to the literature on which it builds. One of the main results is establishing the relation between the LQ stochastic integral game aforementioned and backward stochastic Volterra integral equations (BSVIEs for short), which is based on the results in [3], nonetheless in a much more general setting. First the two-player nature of the game problem demands more delicate manipulations of all the involved Hilbert operators. Secondly we solve a problem left by Chen and Yong in [3] since here the term  $GX^2(T)$  is allowed to appear in the cost functional (4). As a consequence, our paper naturally relates to the BSVIEs theoretic literature. The unique solvability of BSVIEs was firstly studied by Lin [12], see also [24] for later related research. As to the following general form,

$$Y(t) = \psi(t) + \int_t^T g(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \quad (2)$$

we should mention the contribution of Yong ([26], [27]), who firstly introduced the notion of adapted M-solution in [27] and established a Pontryagin type maximum principle for optimal control of stochastic Volterra integral equations with the help of M-solution. Moreover, as shown by Yong in [28], a class of continuous-time dynamic convex and coherent risk measures can be derived via certain BSVIEs. Along this we also refer the reader to [18], [23] for some other studies on this topic.

In a deterministic setting, the game problem for an input-output system governed by Volterra integral equation of the form

$$X(t) = \varphi(t) + \int_0^t [B_1(t, s)u_1(s) + C_1(t, s)u_2(s)]ds$$

with respect to a quadratic performance functional of the form

$$J(u_1(\cdot), u_2(\cdot)) = E \int_0^T [QX^2(t) + R_1u_1^2(t) + R_2u_2^2(t)]dt + EGX^2(T),$$

was studied in [29]. In this context,  $X, u_1$  and  $u_2$  are deterministic functions. With this interpretation, our problem here can be seen as a natural extension of the work in [29]. To our knowledge, the paper might be the first one for the stochastic quadratic integral game. As to the causal feedback optimal control problem for deterministic Volterra integral equation, we would like to mention the works in [29] and [17]. A new method called *projection causality* to this problem was introduced. So when the system is stochastic Volterra integral equation, how to provide a causal feedback implementation of the optimal strategies is still a problem and we hope to study it in the future.

In the traditional stochastic differential game, coupled forward-backward stochastic differential equations (FBSDEs for short) play an important role in the existence of the open-loop saddle points, see for example [15], [25] and the reference cited therein. As to the solvability of coupled FBSDEs, there have been burgeoning research interest in it, see [1], [9], [14], whereas almost all the methods depend heavily on Itô formula or the time-consistent (or semi-group) property of differential equation. In our framework, we will obtain the existence of an open loop saddle point of the quadratic integral game, which is equivalent to the solvability of certain FBSVIE plus the convexity and concavity of the cost functional below. Thus the solvability of coupled forward-backward stochastic Volterra integral equations (FBSVIEs for short) should also play an important role for the stochastic integral game we are tackling. However, the solvability for FBSVIEs is more challenging as compared with the situations for FBSDEs aforementioned. On the one hand, many conventional and convenient approaches or conditions, such as the four-steps method in [14], the monotonicity condition in [9], especially the most important Itô formula, all are absent in this case. On the other hand, because of the lack of time-consistent property for BSVIEs (or FBSVIEs), we can not use the induction directly as differential equation and more complicated things should be involved, see the existence and uniqueness of M-solution of BSVIEs in [27] for detailed accounts. Worse still, the coupling of these two factors greatly amplifies the difficulty of the problem.

In this paper, given assumptions, we will establish the existence and uniqueness of M-solution for coupled FBSVIEs (31), which will be involved in our game problem. As mentioned earlier most of effective techniques in tackling the problem for differential equation become failure, therefore, we have to carry out investigation from some other basic and original views. By assuming that  $\beta$  is a constant, we can introduce a new

equivalent norm for M-solutions of FBSVIEs as follows

$$\begin{aligned} & \| (x(\cdot), y(\cdot), z(\cdot, \cdot)) \|_{\mathcal{H}^2[0,T] \times L^2_{\mathcal{F}}[0,T]}^2 \\ &= E \left[ \int_0^T e^{-\beta s} |x(s)|^2 ds + \int_0^T e^{\beta s} |y(s)|^2 ds + \int_0^T e^{\beta t} \int_0^T |z(t, s)|^2 ds dt \right], \end{aligned}$$

thereby study the unique existence by means of fixed point theorem. This is a common trick employed in the conventional BSDEs case, which also enables us to get around the inapplicability of Itô formula in the current setting. It is also worthy to claim that we do not need more assumptions, such as the monotonicity condition in [9], except the Lipschitz condition. Thus this can be seen as another contribution of this paper.

Notice that as to general form of coupled FBSVIEs, see (32) below, there is hitherto no well technology to deal with, and it is an object of endeavor for us in the future. One substantial difficulty, we believe, was caused by the appearance of  $X(T)$  in the two backward equations of (32). However, in certain special case, we can transform the unique solvability of FBSVIE (32) into the solvability of some kind of backward stochastic Fredholm-Volterra integral equation (BSFVIE for short), allowing the appearance of  $X(T)$ . More importantly, under some assumptions, the aforementioned BSFVIE becomes a forward stochastic Fredholm-Volterra integral equation (SFVIE for short), and this helps us to characterize the Nash equilibrium strategy more explicitly. We refer the readers to [20] and [22] for details on the solvability of SFVIEs. As to the case for BSFVIE, the problem is much more complicated and we hope to study it in the future. At last we will illustrate the application of the obtained results to the stochastic quadratic differential game with delay.

The reminder of this paper is organized as follows. In the next section, the game problem will be formulated and some preliminary results will also be stated. In Section 3 we study the LQ integral games in Hilbert space and obtain one necessary and sufficient condition of existence of saddle point. In Section 4 we will make use of BSVIEs (or FBSVIEs) to characterize the result derived in Section 3 more explicitly. In Section 5 we give another sufficient condition with the help of the solvability of M-solution for coupled FBSVIE (26). Some further considerations, such as the relationship between coupled FBSVIEs, BSFVIEs and SFVIEs are investigated. At last we give some results of stochastic differential game with delay and obtain one explicit expression of the saddle point, which is consistent with the result in [29].

## 2 Problem formulation and preliminary

Let  $(W_t)_{t \in [0, T]}$  be a scalar-valued Wiener process defined on a probability space  $(\Omega, \mathcal{F}, P)$  and  $(\mathcal{F}_t)_{t \in [0, T]}$  denotes the natural filtration of  $(W_t)$ , such that  $\mathcal{F}_0$  contains all  $P$ -null sets of  $\mathcal{F}$ . Our assumption that  $W(\cdot)$  is scalar-valued is for the sake of simplicity and no essential difficulties are encountered when extending our analysis to the case of vector-valued Brownian motion.

Suppose the dynamic of a stochastic system is described by a controlled linear stochastic Volterra integral equation (SVIE for short),

$$\begin{aligned} X(t) = & \varphi(t) + \int_0^t [A_1(t, s)X(s) + B_1(t, s)u_1(s) + C_1(t, s)u_2(s)]ds \\ & + \int_0^t [A_2(t, s)X(s) + B_2(t, s)u_1(s) + C_2(t, s)u_2(s)]dW(s), \end{aligned} \quad (3)$$

where  $u_1$  and  $u_2$  are adapted and stand for, respectively the intervention functions of two agents Play 1 and Play 2 on the dynamic system.  $X$  is the state process and  $u_1$  and  $u_2$  are control processes taken by two players. To avoid undue technicality, we assume both the state process and control process are scalar-valued. We define the cost functional associated with (3) for the players as follows:

$$J(u_1(\cdot), u_2(\cdot)) = E \int_0^T f(t, X(t), u_1(t), u_2(t))dt + EGX^2(T),$$

where

$$\begin{aligned} & f(t, X(t), u_1(t), u_2(t)) \\ = & Q(t)X^2(t) + 2S_1(t)X(t)u_1(t) + 2S_2(t)X(t)u_2(t) + R_{11}(t)u_1^2(t) \\ & + R_{12}(t)u_1(t)u_2(t) + R_{21}(t)u_1(t)u_2(t) + R_{22}(t)u_2^2(t) \\ = & \left\langle \begin{pmatrix} Q(t) & S_1(t) & S_2(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} X(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ u_1(t) \\ u_2(t) \end{pmatrix} \right\rangle_2. \end{aligned}$$

Note that  $\langle \cdot, \cdot \rangle_2$  is defined below. In what follows, we will denote

$$S(t) = \begin{pmatrix} S_1(t) \\ S_2(t) \end{pmatrix}, \quad R(t) = \begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{21}(t) & R_{22}(t) \end{bmatrix}, \quad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix},$$

and

$$\begin{aligned} & J(u_1(\cdot), u_2(\cdot)) \\ = & E \int_0^T [Q(t)X^2(t) + 2X(t)S(t) \cdot u(t) + R(t)u(t) \cdot u(t)]dt + EGX^2(T) \\ = & \langle QX, X \rangle_2 + 2 \langle XS, u \rangle_2 + \langle Ru, u \rangle_2 + E \langle GX(T), X(T) \rangle_1. \end{aligned} \quad (4)$$

Throughout this paper, we assume that  $Q$ ,  $R_{ij}$  and  $S_i$  ( $i, j = 1, 2$ ) are bounded adapted processes and  $G$  is a bounded random variable. Note that in the above model, the controls are allowed to appear in both the drift and diffusion of the state equation, the weighting matrices in the payoff/cost functional are not assumed to be definite/non-singular, and the cross-terms between two controls are allowed to appear, we refer this problem as a so-called zero-sum linear quadratic stochastic integral game.

Next we will give some notations. We denote  $\Delta^c = \{t, s\} \in [0, T]^2; t \leq s\}$  and  $\Delta = [0, T]^2 \setminus \Delta^c$ . Let  $L^2(\Omega \times [0, T])$  be the set of the processes  $X : [0, T] \times \Omega \rightarrow R$  which

is  $\mathcal{B}([0, T]) \times \mathcal{F}_T$ -measurable satisfying  $E \int_0^T |X(t)|^2 dt < \infty$ .  $L^2(\Omega)$  is set of random variable  $\xi : \Omega \rightarrow R$  which is  $\mathcal{F}_T$ -measurable satisfying  $E|\xi|^2 < \infty$ , and we denote its inner product by  $\langle \cdot, \cdot \rangle_1$ .  $\forall R, S \in [0, T]$ ,  $L^2_{\mathcal{F}}[R, S]$  is the set of all adapted processes  $X : [R, S] \times \Omega \rightarrow R$  such that  $E \int_R^S |X(t)|^2 dt < \infty$ , and we denote its inner product by  $\langle \cdot, \cdot \rangle_2$ .  $L^2(R, S; L^2_{\mathcal{F}}[R, S])$  be the set of all process  $Z : [R, S]^2 \times \Omega \rightarrow R$  such that for almost all  $t \in [R, S]$ ,  $Z(t, \cdot)$  is  $\mathcal{F}$ -adapted satisfying  $E \int_R^S \int_R^S |Z(t, s)|^2 ds dt < \infty$ . We denote  $\mathcal{H}^2[R, S] = L^2_{\mathcal{F}}[R, S] \times L^2(R, S; L^2_{\mathcal{F}}[R, S])$ .  $L^\infty[0, T]$  is set of deterministic function  $X : [0, T] \times \Omega \rightarrow R$  such that  $\sup_{t \in [0, T]} |X(t)| < \infty$ .  $L^2(0, T; L^\infty[0, T])$  is set of

deterministic function  $X : [0, T]^2 \rightarrow R$  such that for almost  $t \in [0, T]$ ,  $\sup_{s \in [0, T]} |X(t, s)| < \infty$ .  $L^2(0, T; L^2[0, T])$  is set of deterministic function  $X : [0, T]^2 \rightarrow R$  such that for almost  $t \in [0, T]$ ,  $\int_0^T \int_0^T |X(t, s)|^2 ds < \infty$ . As to  $L^\infty_{\mathbb{F}}[0, T]$ ,  $L^\infty(0, T; L^2_{\mathbb{F}}[0, T])$  and  $L^\infty(0, T; L^\infty_{\mathbb{F}}[0, T])$ , we can define them in a similar manner.

The notion of M-solutions of BSVIEs can be expressed as,

**Definition 2.1** *Let  $S \in [0, T]$ . A pair of  $(Y(\cdot), Z(\cdot, \cdot)) \in \mathcal{H}^2[S, T]$  is called an adapted M-solution of BSVIE (2) on  $[S, T]$  if (2) holds in the usual Itô's sense for almost all  $t \in [S, T]$  and in addition, the following holds:*

$$Y(t) = E^{\mathcal{F}_S} Y(t) + \int_S^t Z(t, s) dW(s), \quad t \in [S, T].$$

In [27], the author gave the definition of M-solution of BSVIE in  $\mathcal{H}^2[0, T]$  and proved the following proposition,

**Proposition 2.1** *Let  $g : \Delta^c \times R \times R \times R \times \Omega \rightarrow R$  be  $\mathcal{B}(\Delta^c \times R \times R \times R) \otimes \mathcal{F}_T$ -measurable such that  $s \rightarrow g(t, s, y, z, \zeta)$  is  $\mathbb{F}$ -progressively measurable for all  $(t, y, z, \zeta) \in [0, T] \times R \times R \times R$ , moreover,  $g$  satisfies the Lipschitz conditions,  $\forall y, \bar{y} \in R, z, \bar{z}, \zeta, \bar{\zeta} \in R$ ,*

$$|g(t, s, y, z, \zeta) - g(t, s, \bar{y}, \bar{z}, \bar{\zeta})| \leq L(t, s)(|y - \bar{y}| + |z - \bar{z}| + |\zeta - \bar{\zeta}|),$$

where  $(t, s) \in \Delta^c$ ,  $\Delta^c = \{(t, s) \in [0, T]^2 \mid t \leq s\}$ ,  $L(t, s)$  is a determined non-negative function satisfying  $\sup_{t \in [0, T]} \int_t^T L^{2+\epsilon}(t, s) ds < \infty$ , for some  $\epsilon > 0$ . Then (2) admits a unique M-solution in  $\mathcal{H}^2[0, T]$ .

### 3 Stochastic LQ integral games in Hilbert spaces

In this section the linear quadratic stochastic integral games problem is formulated in Hilbert space. It is important to recognize that the classical LQ stochastic differential games and LQ optimal control problem for FSVIEs can also be treated similarly in infinite dimensional space, see [15] and [3]. We incorporate some useful techniques in

Chen and Yong [3], but investigate the problem in a more general framework. On the one hand, the coefficients, in both state equation and cost functional, are allowed to be random, moreover, the form of cost functional is general, especially allowing the appearance of the term  $GX^2(T)$ . On the other hand, the nature of game problem also demand more delicate analysis of the operators involved. To start with, we need to make some preliminary.

Let  $\mathcal{H}$  be a Hilbert space and  $\Theta : \mathcal{D}(\Theta) \subseteq \mathcal{H} \rightarrow \mathcal{H}$  be a self-adjoint operator, i.e., it is densely defined and closed but not necessarily bounded. We denote  $\mathcal{R}(\Theta)$  and  $\mathcal{N}(\Theta)$  to be the range and kernel of  $\Theta$ , respectively. Since  $\Theta$  is self-adjoint,  $\mathcal{N}(\Theta)^\perp = \overline{\mathcal{R}(\Theta)}$  and we have  $\Theta \left( \mathcal{D}(\Theta) \cap \overline{\mathcal{R}(\Theta)} \right) \subseteq \mathcal{R}(\Theta)$ . Thus under the decomposition  $\mathcal{H} = \mathcal{N}(\Theta) \oplus \overline{\mathcal{R}(\Theta)}$ , we have the following representation for  $\Theta : \Theta = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Theta} \end{pmatrix}$ , where  $\hat{\Theta} : \mathcal{D}(\Theta) \cap \overline{\mathcal{R}(\Theta)} \subseteq \overline{\mathcal{R}(\Theta)} \rightarrow \overline{\mathcal{R}(\Theta)}$  is self-adjoint (again, it is densely defined and closed, but not necessarily bounded, on the Hilbert space  $\overline{\mathcal{R}(\Theta)}$ ). Now we define the pseudo-inverse  $\Theta^\dagger$  by the following:  $\Theta^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Theta}^{-1} \end{pmatrix}$ , with domain

$$\mathcal{D}(\Theta^\dagger) = \mathcal{N}(\Theta) + \mathcal{R}(\Theta) \equiv \{u_0 + u_1 \mid u_0 \in \mathcal{N}(\Theta), u_1 \in \mathcal{R}(\Theta)\} \supseteq \mathcal{R}(\Theta).$$

Let  $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$  with  $\mathcal{H}_1$  and  $\mathcal{H}_2$  being two Hilbert spaces, and we consider a quadratic functional on  $\mathcal{H}$ : for any  $u = (u_1, u_2)$ ,  $v \in \mathcal{H}$ ,

$$\begin{aligned} J(u) &\equiv J(u_1, u_2) = \langle \Theta u, u \rangle + 2 \langle v, u \rangle \\ &\equiv \left\langle \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle \\ &\quad + 2 \left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\rangle. \end{aligned}$$

Here  $\Theta_{ij} : \mathcal{H}_j \rightarrow \mathcal{H}_i$  ( $i, j = 1, 2$ ) is bounded operator,  $\Theta \equiv \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}$  is self-adjoint. We have the proposition as follows:

**Proposition 3.1** *There exists a saddle point  $(\hat{u}_1, \hat{u}_2) \in \mathcal{H}_1 \times \mathcal{H}_2$  for  $(u_1, u_2) \mapsto J(u_1, u_2)$ , that is,*

$$J(\hat{u}_1, u_2) \leq J(\hat{u}_1, \hat{u}_2) \leq J(u_1, \hat{u}_2), \forall (u_1, u_2) \in \mathcal{H}_1 \times \mathcal{H}_2,$$

*if and only if  $v \in \mathcal{R}(\Theta)$  and the following are true:  $\Theta_{11} \geq 0$  (it means that  $\forall u_1 \in \mathcal{H}_1, \langle \Theta_{11} u_1, u_1 \rangle_{\mathcal{H}_1} \geq 0$ ),  $\Theta_{22} \leq 0$  (it means that  $\forall u_2 \in \mathcal{H}_2, \langle \Theta_{22} u_2, u_2 \rangle_{\mathcal{H}_2} \leq 0$ ). In the above case, each saddle point  $\hat{u} = (\hat{u}_1, \hat{u}_2) \in \mathcal{H}_1 \times \mathcal{H}_2$  is a solution of the equation:  $\Theta \hat{u} + v = 0$ , and it admits a representation:  $\hat{u} = -\Theta^\dagger v + (I - \Theta^\dagger \Theta) \tilde{v}$ , for some  $\tilde{v} \in \mathcal{H}$ . Moreover,  $\hat{u}$  is unique if and only if  $\mathcal{N}(\Theta) = \{0\}$ .*

**Proof.** We refer the reader to see the proof in [15] or [3].  $\square$



The above argument indicates that we could discuss the quadratic integral game by using certain Hilbert operators. Before going further, we need the following standing assumptions which is in force in the rest of the paper.

(H1)  $A_1 \in L^\infty(0, T; L^2_{\mathbb{F}}[0, T])$ ,  $A_2(\cdot, \cdot) \in L^\infty(0, T; L^\infty_{\mathbb{F}}[0, T])$ ,  $B_i(t, s)$  and  $C_i(t, s)$  ( $i = 1, 2$ ) also satisfy the similar assumption.

For any  $(X, u_1, u_2) \in L^2_{\mathcal{F}}[0, T] \times L^2_{\mathcal{F}}[0, T] \times L^2_{\mathcal{F}}[0, T]$ , we can define the operators  $\mathcal{A}$ ,  $\mathcal{B}_1$ ,  $\mathcal{C}_1$  from  $L^2_{\mathcal{F}}[0, T]$  to itself as follows:

$$\begin{aligned} (\mathcal{A}X)(t) &= \int_0^t A_1(t, s)X(s)ds + \int_0^t A_2(t, s)X(s)dW(s), \\ (\mathcal{B}_1u_1)(t) &= \int_0^t B_1(t, s)u_1(s)ds + \int_0^t B_2(t, s)u_1(s)dW(s), \\ (\mathcal{C}_1u_2)(t) &= \int_0^t C_1(t, s)u_2(s)ds + \int_0^t C_2(t, s)u_2(s)dW(s), \end{aligned}$$

thus we have

$$X(t) = \varphi(t) + (\mathcal{A}X)(t) + (\mathcal{B}_1u_1)(t) + (\mathcal{C}_1u_2)(t).$$

The following lemma character the well property of the operators defined above.

**Lemma 3.1** *Let (H1) hold, then the operators  $\mathcal{A}$ ,  $\mathcal{B}_1$  and  $\mathcal{C}_1$  are bounded operators and  $\mathcal{A}$  is quasi-nilpotent, i.e.,  $\overline{\lim}_{k \rightarrow \infty} \|\mathcal{A}^k\|^{\frac{1}{k}} = 0$ . Consequently,  $(I - \mathcal{A})^{-1} : L^2_{\mathcal{F}}[0, T] \rightarrow L^2_{\mathcal{F}}[0, T]$  is bounded, hence, for any  $\varphi(\cdot) \in L^2_{\mathcal{F}}[0, T]$  and  $u_1, u_2 \in L^2_{\mathcal{F}}[0, T]$ , (3) admits a unique solution  $X = (I - \mathcal{A})^{-1}(\varphi + \mathcal{B}_1u_1 + \mathcal{C}_1u_2)$ .*

**Proof.** The proof is essentially resembles the one in [3] and we omit it here.  $\square$

Due to the appearance of  $GX^2(T)$ , some other operators are also required to tackle it. We denote

$$\begin{aligned} \Delta_T X &= \int_0^T A_1(T, s)X(s)ds + \int_0^T A_2(T, s)X(s)dW(s), \\ \Lambda_T u_1 &= \int_0^T B_1(T, s)u_1(s)ds + \int_0^T B_2(T, s)u_1(s)dW(s), \\ \Pi_T u_2 &= \int_0^T C_1(T, s)u_2(s)ds + \int_0^T C_2(T, s)u_2(s)dW(s), \end{aligned}$$

hence

$$X(T) = \Delta_T X + \Lambda_T u_1 + \Pi_T u_2 + \varphi(T).$$

Obviously  $\Delta_T$ ,  $\Lambda_T$  and  $\Pi_T$  are bounded operators from  $L^2_{\mathcal{F}}[0, T]$  to  $L^2(\Omega)$ .

In what follows, we make some conventions as,

$$(\mathcal{U}u)(t) = (\mathcal{B}_1u_1)(t) + (\mathcal{C}_1u_2)(t), \quad \Gamma_T u = \Lambda_T u_1 + \Pi_T u_2,$$



therefore,

$$\mathcal{U}u = (\mathcal{B}_1, \mathcal{C}_1) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (\mathcal{B}_1, \mathcal{C}_1)u, \quad (5)$$

$$\Gamma_T u = (\Lambda_T, \Pi_T) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (\Lambda_T, \Pi_T)u. \quad (6)$$

We define the operators  $\mathcal{Q}$ ,  $\mathcal{S}$  and  $\mathcal{R}$  as follows: for  $i, j = 1, 2$ ,

$$\begin{aligned} \langle \mathcal{Q}X, X \rangle_2 &= E \int_0^T Q(t)X^2(t)dt, & \langle \mathcal{S}X, u \rangle_2 &= E \int_0^T S(t)X(t) \cdot u(t)dt, \\ \langle \mathcal{R}u, u \rangle_2 &= E \int_0^T R(t)u(t) \cdot u(t)dt, & \langle \mathcal{S}_i X, u_i \rangle_2 &= E \int_0^T S_i(t)X(t)u_i(t)dt, \\ \langle \mathcal{R}_{i,j}u_i, u_j \rangle_2 &= E \int_0^T R_{i,j}(t)u_i(t) \cdot u_j(t)dt, \end{aligned}$$

consequently,

$$\mathcal{S} = \begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} \\ \mathcal{R}_{21} & \mathcal{R}_{22} \end{pmatrix}, \quad (7)$$

and (4) can be rewritten as

$$J(u) = \langle \mathcal{Q}X, X \rangle_2 + 2 \langle \mathcal{S}X, u \rangle_2 + \langle \mathcal{R}u, u \rangle_2 + \langle GX(T), X(T) \rangle_1. \quad (8)$$

Now we turn to deal with  $\langle GX(T), X(T) \rangle_1$  by means of the operators defined previously.

$$\begin{aligned} &\langle GX(T), X(T) \rangle_1 \\ &= \langle G(\Delta_T X + \Gamma_T u + \varphi(T)), \Delta_T X + \Gamma_T u + \varphi(T) \rangle_1 \\ &= \langle \Delta_T^* G \Delta_T X, X \rangle_2 + 2 \langle \Gamma_T^* G \Delta_T X, u \rangle_2 + \langle \Gamma_T^* G \Gamma_T u, u \rangle_2 \\ &\quad + 2 \langle X, \Delta_T^* G \varphi(T) \rangle_2 + 2 \langle u, \Gamma_T^* G \varphi(T) \rangle_2 + \langle \varphi(T), \varphi(T) \rangle_1, \end{aligned} \quad (9)$$

where  $\forall \eta \in L^2(\Omega)$ ,  $X, u \in L^2_{\mathcal{F}}[0, T]$ ,

$$\langle \Delta_T X, \eta \rangle_1 = \langle X, \Delta_T^* \eta \rangle_2, \quad \langle \Gamma_T u, \eta \rangle_1 = \langle u, \Gamma_T^* \eta \rangle_2.$$

If we denote

$$\mathcal{Q}' = \mathcal{Q} + \Delta_T^* G \Delta_T, \quad \mathcal{S}' = \mathcal{S} + \Gamma_T^* G \Delta_T, \quad \mathcal{R}' = \mathcal{R} + \Gamma_T^* G \Gamma_T, \quad (10)$$

then we can obtain the following expressions after substituting (9) into (8),

$$\begin{aligned}
J(u) &= \langle \mathcal{Q}'X, X \rangle_2 + 2\langle \mathcal{S}'X, u \rangle_2 + \langle \mathcal{R}'u, u \rangle_2 \\
&\quad + 2\langle X, \Delta_T^* G\varphi(T) \rangle_2 + 2\langle u, \Gamma_T^* G\varphi(T) \rangle_2 + \langle \varphi(T), \varphi(T) \rangle_1 \\
&= \left\langle \begin{pmatrix} \mathcal{Q}' & \mathcal{S}'^* \\ \mathcal{S}' & \mathcal{R}' \end{pmatrix} \begin{pmatrix} (I - \mathcal{A})^{-1}(\varphi + \mathcal{U}u) \\ u \end{pmatrix}, \begin{pmatrix} (I - \mathcal{A})^{-1}(\varphi + \mathcal{U}u) \\ u \end{pmatrix} \right\rangle_2 \\
&\quad + 2\langle (I - \mathcal{A})^{-1}(\varphi + \mathcal{U}u), \Delta_T^* G\varphi(T) \rangle_2 + 2\langle u, \Gamma_T^* G\varphi(T) \rangle_2 + \langle \varphi(T), \varphi(T) \rangle_1 \\
&= \left\langle \begin{pmatrix} \mathcal{Q}' & \mathcal{S}'^{*T} \\ \mathcal{S}' & \mathcal{R}' \end{pmatrix} \begin{pmatrix} (I - \mathcal{A})^{-1} & (I - \mathcal{A})^{-1}\mathcal{U} \\ 0 & I \end{pmatrix} \begin{pmatrix} \varphi \\ u \end{pmatrix}, \right. \\
&\quad \left. \begin{pmatrix} (I - \mathcal{A})^{-1} & (I - \mathcal{A})^{-1}\mathcal{U} \\ 0 & I \end{pmatrix} \begin{pmatrix} \varphi \\ u \end{pmatrix} \right\rangle_2 \\
&\quad + 2\langle u, \mathcal{U}^*(I - \mathcal{A}^*)^{-1}\Delta_T^* G\varphi(T) + \Gamma_T^* G\varphi(T) \rangle_2 \\
&\quad + 2\langle (I - \mathcal{A})^{-1}\varphi, \Delta_T^* G\varphi(T) \rangle + \langle \varphi(T), \varphi(T) \rangle_1 \\
&= \langle \Theta u, u \rangle_2 + \langle \Theta_1 \varphi, u \rangle_2 + \langle \Theta_2 \varphi, \varphi \rangle_2 \\
&\quad + 2\langle (I - \mathcal{A})^{-1}\varphi, \Delta_T^* G\varphi(T) \rangle + \langle \varphi(T), \varphi(T) \rangle_1,
\end{aligned}$$

where

$$\begin{aligned}
\Theta &= (\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}\mathcal{Q}' + \mathcal{S}')(I - \mathcal{A})^{-1}\mathcal{U} + \mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}\mathcal{S}'^{*T} + \mathcal{R}', \\
\Theta_1 \varphi &= (\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}\mathcal{Q}' + \mathcal{S}')(I - \mathcal{A})^{-1}\varphi + \mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}\Delta_T^* G\varphi(T) + \Gamma_T^* G\varphi(T), \\
\Theta_2 &= (I - \mathcal{A}^*)^{-1}\mathcal{Q}'(I - \mathcal{A})^{-1}.
\end{aligned} \tag{11}$$

In above,  $A^T$  is the transpose of  $A$ . From (5), (6) and (10) we have

$$\begin{aligned}
&(\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}\mathcal{Q}' + \mathcal{S}')(I - \mathcal{A})^{-1}\mathcal{U} \\
&= \begin{pmatrix} \mathcal{B}_1^* \\ \mathcal{C}_1^* \end{pmatrix} (I - \mathcal{A}^*)^{-1}\mathcal{Q}'(I - \mathcal{A})^{-1}(\mathcal{B}_1, \mathcal{C}_1) \\
&\quad + \begin{pmatrix} \mathcal{S}_1 \\ \mathcal{S}_2 \end{pmatrix} (I - \mathcal{A})^{-1}(\mathcal{B}_1, \mathcal{C}_1) + \begin{pmatrix} \Lambda_T^* \\ \Pi_T^* \end{pmatrix} G\Delta_T(I - \mathcal{A})^{-1}(\mathcal{B}_1, \mathcal{C}_1),
\end{aligned}$$

and

$$\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}\mathcal{S}'^{*T} = \begin{pmatrix} \mathcal{B}_1^* \\ \mathcal{C}_1^* \end{pmatrix} (I - \mathcal{A}^*)^{-1}[(\mathcal{S}_1^*, \mathcal{S}_2^*) + \Delta_T^* G(\Lambda_T, \Pi_T)],$$

so we have

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix},$$

where

$$\begin{aligned}
\Theta_{11} &= \mathcal{B}_1^*(I - \mathcal{A}^*)^{-1}\mathcal{Q}'(I - \mathcal{A})^{-1}\mathcal{B}_1 + \mathcal{S}_1(I - \mathcal{A})^{-1}\mathcal{B}_1 + \Lambda_T^* G\Delta_T(I - \mathcal{A})^{-1}\mathcal{B}_1 \\
&\quad + \mathcal{B}_1^*(I - \mathcal{A}^*)^{-1}(\mathcal{S}_1^* + \Delta_T^* G\Lambda_T) + \mathcal{R}_{11} + \Lambda_T^* G\Lambda_T,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}\Theta_{22} = & \mathcal{C}_1^*(I - \mathcal{A}^*)^{-1}\mathcal{Q}'(I - \mathcal{A})^{-1}\mathcal{C}_1 + \mathcal{S}_2(I - \mathcal{A})^{-1}\mathcal{C}_1 + \Pi_T^*G\Delta_T(I - \mathcal{A})^{-1}\mathcal{C}_1 \\ & + \mathcal{C}_1^*(I - \mathcal{A}^*)^{-1}(\mathcal{S}_2^* + \Delta_T^*G\Pi_T) + \mathcal{R}_{22} + \Pi_T^*G\Pi_T.\end{aligned}\quad (13)$$

To conclude this section, we state a necessary and sufficient condition of existence of saddle point for open-loop game with the help of Proposition 3.1 and the operators above.

**Theorem 3.1** *Let (H1) hold, for given  $\varphi(\cdot) \in L_{\mathcal{F}}^2[0, T]$ , the open-loop game admits a saddle point  $\hat{u} \equiv (\hat{u}_1, \hat{u}_2)$  if and only if  $\Theta_{11} \geq 0$ ,  $\Theta_{22} \leq 0$  and  $\Theta_1\varphi \in \mathcal{R}(\Theta)$ , where  $\Theta_{11}$  and  $\Theta_{22}$  are defined by (12) and (13). In this case, any saddle point  $\hat{u}$  is a solution of the following equation:  $\Theta u + \Theta_1\varphi = 0$  with  $\Theta_1\varphi$  defined in (11), and it admits the following representation:*

$$\hat{u} = -\Theta^\dagger\Theta_1\varphi + (I - \Theta^\dagger\Theta)v,$$

for some  $v \in L_{\mathcal{F}}^2[0, T] \times L_{\mathcal{F}}^2[0, T]$ . In addition, the saddle point is unique if and only if  $\mathcal{N}(\Theta) = \{0\}$ .

The proof is obvious and we omit it here. Note that here  $\Theta_{11} \geq 0$ ,  $\Theta_{22} \leq 0$  is equivalent to the convexity of  $u_1 \mapsto J_0(u_1, 0)$  and the concavity of  $u_2 \mapsto J_0(0, u_2)$ , where  $J_0(u)$  is the value of  $J(u)$  when  $\varphi \equiv 0$ .

## 4 Open-loop games via BSVIE

In this section, to further characterize explicitly the sufficient and necessary condition in Theorem 3.1, we will make use of an efficient tool, i.e., BSVIEs aforementioned. Two equivalent conditions correspondent to the one in Theorem 3.1 are proposed and analyzed via BSVIEs. The method is designed around the scheme in [3] but with some more delicate and sophisticated analysis involved. At the outset, we need to prove some lemmas needed in the sequel.

**Lemma 4.1** *Let (H1) hold. Then for any  $\rho(\cdot) \in L_{\mathcal{F}}^2[0, T]$ ,  $(\mathcal{A}^*\rho)(t) = \sigma(t)$ ,  $t \in [0, T]$ , where*

$$\begin{aligned}\sigma(t) &= E^{\mathcal{F}_t} \int_t^T [A_1(s, t)\rho(s) + A_2(s, t)\nu(s, t)]ds, \\ \rho(t) &= E\rho(t) + \int_0^t \nu(t, s)dW(s), \quad t \in [0, T].\end{aligned}\quad (14)$$

Similarly we have  $\forall u_i \in L_{\mathcal{F}}^2[0, T]$ , ( $i = 1, 2$ ),  $(\mathcal{B}_1^*u_1)(t) = \alpha(t)$ ,  $(\mathcal{C}_1^*u_2)(t) = \gamma(t)$ ,  $t \in [0, T]$ , where

$$\begin{aligned}\alpha(t) &= E^{\mathcal{F}_t} \int_t^T [B_1(s, t)u_1(s) + B_2(s, t)\beta(s, t)]ds, \\ u_1(t) &= Eu_1(t) + \int_0^t \beta(t, s)dW(s), \quad t \in [0, T],\end{aligned}$$

and

$$\begin{aligned}\gamma(t) &= E^{\mathcal{F}_t} \int_t^T [C_1(s, t)u_2(s) + C_2(s, t)\delta(s, t)]ds, \\ u_2(t) &= Eu_2(t) + \int_0^t \delta(t, s)dW(s), \quad t \in [0, T].\end{aligned}$$

**Proof.** Since  $\mathcal{A}$  is a bounded linear operator from the Hilbert space  $L^2_{\mathcal{F}}[0, T]$  into itself, thus the adjoint operator  $\mathcal{A}^*$  of  $\mathcal{A}$  is well-defined. For any  $X(\cdot) \in L^2_{\mathcal{F}}[0, T]$ ,

$$\begin{aligned}& E \int_0^T (\mathcal{A}^* \rho)(t)X(t)dt \equiv E \int_0^T \rho(t)(\mathcal{A}X)(t)dt \\ &= E \int_0^T \rho(t)dt \int_0^t A_1(t, s)X(s)ds + E \int_0^T \rho(t)dt \int_0^t A_2(t, s)X(s)dW(s) \\ &= E \int_0^T X(t)dt \int_t^T A_1(s, t)\rho(s)ds + E \int_0^T X(t)dt \int_t^T A_2(s, t)\nu(s, t)ds \\ &= E \int_0^T X(t)dt E^{\mathcal{F}_t} \int_t^T [A_1(s, t)\rho(s) + A_2(s, t)\nu(s, t)]ds,\end{aligned}$$

where we use the relation  $\rho(t) = E\rho(t) + \int_0^t \nu(t, s)dW(s)$  and stochastic Fubini theorem above, thus by the arbitrariness of  $X$ , we get (14). As to the other two results, the proof is similar.  $\square$

**Remark 4.1** Let us consider the following equation:

$$Y(t) = \psi(t) + \int_t^T [A_1(s, t)Y(s) + A_2(s, t)Z(s, t)]ds - \int_t^T Z(t, s)dW(s), \quad (15)$$

where  $(Y(\cdot), Z(\cdot, \cdot))$  is the unique  $M$ -solution of (15) and  $\psi \in L^2(\Omega \times [0, T])$ . From Lemma 4.1, we know that  $Y = E^{\mathcal{F}_t}\psi + \mathcal{A}^*Y$ . Since  $(I - \mathcal{A})^{-1}$  exists and bounded, we have  $Y = (I - \mathcal{A}^*)^{-1}E^{\mathcal{F}_t}\psi$ .

**Lemma 4.2** Let (H1) hold. Then  $\forall \eta \in L^2(\Omega)$ , we have

$$\begin{aligned}(\Delta_T^* \eta)(s) &= A_1(T, s)E^{\mathcal{F}_s}\eta + A_2(T, s)\theta(s), \\ \eta &= E\eta + \int_0^T \theta(s)dW(s), \quad t \in [0, T].\end{aligned} \quad (16)$$

Similarly  $\forall \zeta_i \in L^2(\Omega)$ , ( $i = 1, 2$ ), we have

$$\begin{aligned}(\Lambda_T^* \zeta_1)(s) &= B_1(T, s)E^{\mathcal{F}_s}\zeta_1 + B_2(T, s)\kappa_1(s), \\ \zeta_1 &= E\zeta_1 + \int_0^T \kappa_1(s)dW(s), \quad t \in [0, T],\end{aligned}$$

and

$$\begin{aligned}(\Pi_T^* \zeta_2)(s) &= C_1(T, s)E^{\mathcal{F}_s}\zeta_2 + C_2(T, s)\kappa_2(s), \\ \zeta_2 &= E\zeta_2 + \int_0^T \kappa_2(s)dW(s), \quad t \in [0, T].\end{aligned}$$

**Proof.** Because  $\Delta$  is a bounded linear operator from the Hilbert space  $L^2_{\mathcal{F}}[0, T]$  into  $L^2(\Omega)$ , thus the adjoint operator  $\Delta^*$  of  $\Delta$ , which is defined from  $L^2(\Omega)$  into  $L^2_{\mathcal{F}}[0, T]$ , is well-defined. For any  $\eta \in L^2(\Omega)$ ,  $X \in L^2_{\mathcal{F}}[0, T]$ , we have

$$\begin{aligned}
E \int_0^T (\Delta_T^* \eta)(s) X(s) ds &= \langle \Delta_T^* \eta, X \rangle_2 = \langle \eta, \Delta_T X \rangle_1 = E \eta \Delta_T X \\
&= E \int_0^T A_1(T, s) \eta X(s) ds + E \int_0^T A_2(T, s) X(s) \eta dW(s) \\
&= E \int_0^T A_1(T, s) \eta X(s) ds + E \int_0^T A_2(T, s) \theta(s) X(s) ds \\
&= E \int_0^T [A_1(T, s) \eta + A_2(T, s) \theta(s)] X(s) ds \\
&= E \int_0^T [A_1(T, s) E^{\mathcal{F}_s} \eta + A_2(T, s) \theta(s)] X(s) ds. \tag{17}
\end{aligned}$$

Since  $X(\cdot) \in L^2_{\mathcal{F}}[0, T]$  is arbitrary, it follows from (17) that,

$$(\Delta_T^* \eta)(s) = A_1(T, s) E^{\mathcal{F}_s} \eta + A_2(T, s) \theta(s).$$

As to the others, the proof is similar.  $\square$

The previous two lemmas show the way to express the Hilbert operators more clearly. The following two theorems are the two main results in this section, which are established with the help of the two lemmas above.

**Theorem 4.1** *Let (H1) hold, then for  $i = 1, 2$ , and any  $u_i(\cdot) \in L^2_{\mathcal{F}}[0, T]$ ,  $(X^{u_1}, Y^{u_1}, Z^{u_1}, \lambda^{u_1})$  is the unique M-solution of the following decoupled FBSVIE:*

$$\left\{ \begin{aligned} X^{u_1}(t) &= \int_0^t [A_1(t, s) X^{u_1}(s) + B_1(t, s) u_1(s)] ds \\ &\quad + \int_0^t [A_2(t, s) X^{u_1}(s) + B_2(t, s) u_1(s)] dW(s), \\ Y^{u_1}(t) &= Q(t) X^{u_1}(t) + S_1(t) u_1(t) + A_1(T, t) G X^{u_1}(T) + A_2(T, t) \theta_1(t) \\ &\quad + \int_t^T [A_1(s, t) Y^{u_1}(s) + A_2(s, t) Z^{u_1}(s, t)] ds - \int_t^T Z^{u_1}(t, s) dW(s), \\ \lambda^{u_1}(t) &= E^{\mathcal{F}_t} \int_t^T [B_1(s, t) Y^{u_1}(s) + B_2(s, t) Z^{u_1}(s, t)] ds, \end{aligned} \right. \tag{18}$$

and  $(X^{u_2}, Y^{u_2}, Z^{u_2}, \lambda^{u_2})$  is the unique M-solution of the following decoupled FBSVIE:

$$\left\{ \begin{aligned} X^{u_2}(t) &= \int_0^t [A_1(t, s) X^{u_2}(s) + C_1(t, s) u_2(s)] ds \\ &\quad + \int_0^t [A_2(t, s) X^{u_2}(s) + C_2(t, s) u_2(s)] dW(s), \\ Y^{u_2}(t) &= Q(t) X^{u_2}(t) + S_2(t) u_2(t) + A_1(T, t) G X^{u_2}(T) + A_2(T, t) \theta_2(t) \\ &\quad + \int_t^T [A_1(s, t) Y^{u_2}(s) + A_2(s, t) Z^{u_2}(s, t)] ds - \int_t^T Z^{u_2}(t, s) dW(s), \\ \lambda^{u_2}(t) &= E^{\mathcal{F}_t} \int_t^T [C_1(s, t) Y^{u_1}(s) + C_2(s, t) Z^{u_2}(s, t)] ds, \end{aligned} \right. \tag{19}$$

where  $i = 1, 2$ ,

$$GX^{u_i}(T) = EGX^{u_i}(T) + \int_0^T \theta_i(s) dW(s).$$

Then  $\Theta_{11} \geq 0$  is equivalent to:  $\forall u_1(t) \in L^2_{\mathcal{F}}[0, T]$ ,

$$\begin{aligned} & E \int_0^T [\lambda^{u_1}(s) + S_1(s)X^{u_1}(s) + R_{11}(s)u_1(s)]u_1(s)ds \\ & + E \int_0^T [B_1(T, s)E^{\mathcal{F}_s}GX^{u_1}(T) + B_2(T, s)\theta_1(s)]u_1(s)ds \geq 0, \end{aligned} \quad (20)$$

and  $\Theta_{22} \leq 0$  is equivalent to:  $\forall u_2(t) \in L^2_{\mathcal{F}}[0, T]$ ,

$$\begin{aligned} & E \int_0^T [\lambda^{u_2}(s) + S_2(s)X^{u_2}(s) + R_{22}(s)u_2(s)]u_2(s)ds \\ & + E \int_0^T [C_1(T, s)E^{\mathcal{F}_s}GX^{u_2}(T) + C_2(T, s)\theta_2(s)]u_2(s)ds \leq 0. \end{aligned} \quad (21)$$

**Proof.** It is clear that  $\forall u_1(t) \in L^2_{\mathcal{F}}[0, T]$ ,

$$\begin{aligned} & \mathcal{B}_1^*(I - \mathcal{A}^*)^{-1}(\mathcal{Q}'(I - \mathcal{A})^{-1}\mathcal{B}_1u_1 + \mathcal{S}_1^*u_1 + \Delta_T^*G\Lambda_Tu_1) \\ & = \mathcal{B}_1^*(I - \mathcal{A}^*)^{-1}[(\mathcal{Q} + \Delta_T^*G\Delta_T)(I - \mathcal{A})^{-1}\mathcal{B}_1u_1 + \mathcal{S}_1^*u_1 + \Delta_T^*G\Lambda_Tu_1] \\ & = \mathcal{B}_1^*(I - \mathcal{A}^*)^{-1}(QX^{u_1} + S_1u_1 + \Delta_T^*GX^{u_1}(T)), \end{aligned}$$

and

$$\begin{aligned} & S_1(I - \mathcal{A})^{-1}\mathcal{B}_1u_1 + \mathcal{R}_{11}u_1 + \Lambda_T^*G\Delta_T(I - \mathcal{A})^{-1}\mathcal{B}_1u_1 + \Lambda_T^*G\Lambda_Tu_1 \\ & = S_1X^{u_1} + R_{11}u_1 + \Lambda_T^*GX^{u_1}(T), \end{aligned}$$

where  $X^{u_1}(t)$  and  $X^{u_1}(T)$  can be expressed by

$$\begin{aligned} X^{u_1}(t) &= \int_0^t B_1(t, s)u_1(s)ds + \int_0^t B_2(t, s)u_1(s)dW(s) \\ &\quad + \int_0^t A_1(t, s)X^{u_1}(s)ds + \int_0^t A_2(t, s)X^{u_1}(s)dW(s), \\ X^{u_1}(T) &= \int_0^T B_1(T, s)u_1(s)ds + \int_0^T B_2(T, s)u_1(s)dW(s) \\ &\quad + \int_0^T A_1(T, s)X^{u_1}(s)ds + \int_0^T A_2(T, s)X^{u_1}(s)dW(s), \end{aligned}$$

thereby

$$\begin{aligned} (\Theta_{11}u) &= \mathcal{B}_1^*(I - \mathcal{A}^*)^{-1}[QX^{u_1} + S_1u_1 + A_1(T, \cdot)E^{\mathcal{F}}GX^{u_1}(T) \\ &\quad + A_2(T, \cdot)\theta_1(\cdot)] + S_1X^{u_1} + R_{11}u_1 \\ &\quad + B_1(T, \cdot)E^{\mathcal{F}}GX^{u_1}(T) + B_2(T, \cdot)\theta_1(\cdot), \end{aligned}$$

where

$$GX^{u_1}(T) = EGX^{u_1}(T) + \int_0^T \theta_1(s) dW(s).$$

By Lemma 4.1 and Lemma 4.2 we have

$$\begin{aligned} (\Theta_{11}u)(s) &= \lambda^{u_1}(s) + S_1(s)X^{u_1}(s) + R_{11}(s)u_1(s) \\ &+ B_1(T, s)E^{\mathcal{F}_s}GX^{u_1}(T) + B_2(T, s)\theta_1(s), \end{aligned}$$

where  $(Y^{u_1}, Z^{u_1}, \lambda^{u_1})$  is the unique M-solution of the following BSVIEs,

$$\begin{cases} Y^{u_1}(t) = Q(t)X^{u_1}(t) + S_1(t)u_1(t) + A_1(T, t)E^{\mathcal{F}_t}GX^{u_1}(T) + A_2(T, t)\theta_1(t) \\ \quad + \int_t^T [A_1(s, t)Y^{u_1}(s) + A_2(s, t)Z^{u_1}(s, t)]ds - \int_t^T Z^{u_1}(t, s)dW(s), \\ \lambda^{u_1}(t) = E^{\mathcal{F}_t} \int_t^T [B_1(s, t)Y^{u_1}(s) + B_2(s, t)Z^{u_1}(s, t)]ds. \end{cases} \quad (22)$$

In the similar method we have

$$\begin{aligned} (\Theta_{22}u) &= \mathcal{C}_1^*(I - \mathcal{A}^*)^{-1}[QX^{u_2} + S_2u_2 + \Delta_T^*GX^{u_2}(T)] \\ &\quad + S_2X^{u_2} + R_{22}u_2 + \Pi_T^*GX^{u_2}(T) \\ &= \lambda^{u_2} + S_2X^{u_2} + R_{22}u_2 + C_1(T, s)E^{\mathcal{F}_s}GX^{u_2}(T) + C_2(T, \cdot)\theta_2(\cdot), \end{aligned}$$

and

$$GX^{u_2}(T) = EGX^{u_2}(T) + \int_0^T \theta_2(s) dW(s),$$

where  $(Y^{u_2}, Z^{u_2}, \lambda^{u_2})$  is the unique M-solution of the following BSVIEs,

$$\begin{cases} Y^{u_2}(t) = QX^{u_2}(t) + S_2u_2(t) + A_1(T, t)E^{\mathcal{F}_t}GX^{u_2}(T) + A_2(T, t)\theta_2(t) \\ \quad + \int_t^T [A_1(s, t)Y^{u_2}(s) + A_2(s, t)Z^{u_2}(s, t)]ds - \int_t^T Z^{u_2}(t, s)dW(s), \\ \lambda^{u_2}(t) = E^{\mathcal{F}_t} \int_t^T [C_1(s, t)Y^{u_2}(s) + C_2(s, t)Z^{u_2}(s, t)]ds. \end{cases}$$

Hence the conclusion hold naturally.  $\square$

Note that (18) or (19) admits a unique M-solution  $(X^{u_i}, Y^{u_i}, Z^{u_i}, \lambda^{u_i})$  by which we mean that  $(Y^{u_i}, Z^{u_i})$  is the unique M-solution of the second BSVIE and  $(X^{u_i}, \lambda^{u_i})$  is the unique adapted solution of the other two equations.

**Theorem 4.2** *Let (H1) hold,  $\varphi(\cdot) \in L_{\mathcal{F}}^2[0, T]$ , then*

$$(\Theta u)(t) + (\Theta_1\varphi)(t) = \lambda(t) + (SX)(t) + (Ru)(t) + \Xi_1(t)GX(T) + \Xi_2(t)\theta(t),$$

where  $\lambda(\cdot)$  satisfies

$$\lambda(t) = E^{\mathcal{F}_t} \int_t^T \left[ \begin{pmatrix} B_1(s, t) \\ C_1(s, t) \end{pmatrix} Y(s) + \begin{pmatrix} B_2(s, t) \\ C_2(s, t) \end{pmatrix} Z(s, t) \right] ds, \quad (23)$$



$Y(\cdot)$  is the unique  $M$ -solution of BSVIE

$$\begin{aligned} Y(t) = & Q(t)X(t) + S^T(t)u(t) + A_1(T, t)E^{\mathcal{F}_t}GX(T) + A_2(T, t)\theta(t) \\ & + \int_t^T [A_1(s, t)Y(s) + A_2(s, t)Z(s, t)]ds - \int_t^T Z(t, s)dW(s), \end{aligned} \quad (24)$$

$GX(T) = EGX(T) + \int_0^T \theta(s)dW(s)$ , and  $X(t)$  is the unique solution of (3). Consequently, the condition  $\Theta_1\varphi \in \mathcal{R}(\Theta)$  holds if and only if there is a  $\hat{u}(\cdot)$  such that

$$\lambda(t) + (SX)(t) + (R\hat{u})(t) + \Xi_1(t)GX(T) + \Xi_2(t)\theta(t) = 0, \quad (25)$$

where  $\Xi_i$  are defined below.

**Proof.** It follows from (11) that

$$\begin{aligned} (\Theta_1\varphi)(t) &= [\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}\mathcal{Q}'(I - \mathcal{A})^{-1}\varphi](t) + [\mathcal{S}'(I - \mathcal{A})^{-1}\varphi](t) \\ &\quad + [\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}\Delta_T^*G\varphi(T)](t) + [\Gamma_T^*G\varphi(T)](t), \\ &= \mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}[(QX^\varphi)(t) + (\Delta_T^*G\Delta_T X^\varphi)(t) + (\Delta_T^*G\varphi(T))(t)] \\ &\quad + (SX^\varphi)(t) + (\Gamma_T^*G\Delta_T X^\varphi)(t) + (\Gamma_T^*G\varphi(T))(t), \end{aligned}$$

where

$$X^\varphi(t) = \varphi(t) + \int_0^t A_1(t, s)X^\varphi(s)ds + \int_0^t A_2(t, s)X^\varphi(s)dW(s).$$

So we have

$$\begin{aligned} & (\Theta u)(t) + (\Theta_1\varphi)(t) \\ = & [\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}(QX + S^T u)](t) \\ & + [\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}(\Delta_T^*G\Delta_T X + \Delta_T^*G\Gamma_T u + \Delta_T^*G\varphi(T))](t) \\ & + (SX)(t) + (\Gamma_T^*G\Delta_T X)(t) + (Ru)(t) + (\Gamma_T^*G\Gamma_T u)(t) + (\Gamma_T^*G\varphi(T))(t) \\ = & [\mathcal{U}^{*T}(I - \mathcal{A}^*)^{-1}(QX + S^T u + \Delta_T^*GX(T))](t) \\ & + (SX)(t) + (Ru)(t) + \Xi_1(t)E^{\mathcal{F}_t}GX(T) + \Xi_2(t)\theta(t) \\ = & \lambda(t) + (SX)(t) + (Ru)(t) + \Xi_1(t)E^{\mathcal{F}_t}GX(T) + \Xi_2(t)\theta(t), \end{aligned}$$

where

$$\Xi_1(t) = \begin{pmatrix} B_1(T, t) \\ C_1(T, t) \end{pmatrix}, \quad \Xi_2(t) = \begin{pmatrix} B_2(T, t) \\ C_2(T, t) \end{pmatrix},$$

and  $\lambda(\cdot)$  satisfies

$$\lambda(t) = E^{\mathcal{F}_t} \int_t^T \left[ \begin{pmatrix} B_1(s, t) \\ C_1(s, t) \end{pmatrix} Y(s) + \begin{pmatrix} B_2(s, t) \\ C_2(s, t) \end{pmatrix} Z(s, t) \right] ds,$$

where  $Y(\cdot)$  is the  $M$ -solution of BSVIE

$$\begin{aligned} Y(t) = & Q(t)X(t) + S^T(t)u(t) + A_1(T, t)E^{\mathcal{F}_t}GX(T) + A_2(T, t)\theta(t) \\ & + \int_t^T [A_1(s, t)Y(s) + A_2(s, t)Z(s, t)]ds - \int_t^T Z(t, s)dW(s), \end{aligned}$$

and  $\theta(\cdot)$  is determined by

$$GX(T) = EGX(T) + \int_0^T \theta(t) dW(t).$$

Then the conclusion follows.  $\square$

## 5 Stochastic integral games and coupled FBSIVEs

### 5.1 A sufficient condition for existence of saddle point

In this subsection, under certain assumptions, a sufficient condition for the existence of saddle point  $\hat{u}$  will be given via coupled FBSVIEs. To begin with, we should investigate the solvability the following coupled FBSVIE on  $[0, T]$ ,

$$\left\{ \begin{array}{l} X(t) = \varphi(t) + \int_0^t [A_1(t, s)X(s) + B_1(t, s)P(s)]ds \\ \quad + \int_0^t [A_2(t, s)X(s) + B_2(t, s)P(s)]dW(s), \\ Y(t) = \phi_1(t)X(t) + \phi_2(t)P(t) + \int_t^T C_1(s, t)Y(s)ds \\ \quad + \int_t^T C_2(s, t)Z(s, t)ds - \int_t^T Z(t, s)dW(s), \\ P(t) = E^{\mathcal{F}_t} \int_t^T [D_1(s, t)Y(s) + D_2(s, t)Z(s, t)]ds. \end{array} \right. \quad (26)$$

We will show that it admits a unique M-solution by which we mean that  $(X, Y, Z, P)$  satisfies the FBSVIEs in the usual sense and moreover the following hold,  $Y(t) = EY(t) + \int_0^t Z(t, s)dW(s)$ . Note that the generator in the second equation is independent of  $Z(t, s)$  with  $(t, s) \in \Delta^c$ , we can transform the above FBSVIE into another form

$$\left\{ \begin{array}{l} X(t) = \varphi(t) + \int_0^t A_1(t, s)X(s)ds + \int_0^t A_2(t, s)X(s)dW(s) \\ \quad + \int_0^t B_1(t, s) \left( E^{\mathcal{F}_s} \int_s^T [D_1(u, s)Y(u)du + D_2(u, s)Z(u, s)]du \right) ds \\ \quad + \int_0^t B_2(t, s) \left( E^{\mathcal{F}_s} \int_s^T [D_1(u, s)Y(u)du + D_2(u, s)Z(u, s)]du \right) dW(s), \\ Y(t) = \phi_1(t)X(t) + E^{\mathcal{F}_t} \int_t^T C'_1(s, t)Y(s)ds + E^{\mathcal{F}_t} \int_t^T C'_2(s, t)Z(s, t)ds, \end{array} \right. \quad (27)$$

where  $C'_1(s, t) = C_1(s, t) + \phi_2(t)D_1(s, t)$ , and  $C'_2(s, t) = C_2(s, t) + \phi_2(t)D_2(s, t)$ . Next we turn to study FBSVIE (27) rather than (26). Basic assumptions imposed on the parameters in the above equation are summarized as follows,

(H2)  $A_i(B_i) : \Delta \times \Omega \mapsto R$  ( $C_i, D_i : \Delta^c \times \Omega \mapsto R$ , respectively) is  $\mathbb{B}(\Delta) \otimes \mathcal{F}_T$ -measurable ( $\mathbb{B}(\Delta^c) \otimes \mathcal{F}_T$  - measurable, respectively) such that  $s \rightarrow A_i(t, s)(B_i(t, s))$

$(s \rightarrow C_i(s, t)(D_i(s, t)), \text{respectively})$  is  $\mathcal{F}$ - progressively measurable for all  $t \in [0, T]$ ,  $(i = 1, 2)$ ,  $\varphi(\cdot) \in L^2_{\mathcal{F}}[0, T]$ ,  $\phi_i$  ( $i = 1, 2$ ) is deterministic function. We assume that for any  $t \geq s$ ,

$$\begin{aligned} |A_1(t, s)| &\leq K_1(t, s), \quad |A_2(t, s)| \leq K_2(t, s), \\ |B_1(t, s)| &\leq e^{\beta s} K_3(t, s), \quad |B_2(t, s)| \leq e^{\beta s} K_4(t, s), \end{aligned}$$

and for any  $t \leq s$

$$\begin{aligned} |C_1(t, s)| &\leq K_5(t, s), \quad |C_2(t, s)| \leq K_6(t, s), \\ \phi_2(t)|D_1(t, s)| &\leq K_7(t, s), \quad \phi_2(t)|D_2(t, s)| \leq K_8(t, s), \end{aligned}$$

where  $q > 2$  is a constant,  $|\phi_1(t)| \leq \frac{1}{2}e^{-\beta t}$  with  $\beta > 1$  being a constant and

$$\begin{aligned} M_1 &= \sup_{t \in [0, T]} \int_0^t K_1^2(t, s) < \infty, \quad M_3 = \sup_{t \in [0, T]} \int_0^t K_3^2(t, s) < \infty, \\ M_2 &= \sup_{(t, s) \in \Delta} K_2(t, s) < \infty, \quad M_4 = \sup_{(t, s) \in \Delta} K_4(t, s), \\ M_5 &= \sup_{t \in [0, T]} \int_t^T K_5^2(s, t) ds < \infty, \quad M_6 = \sup_{t \in [0, T]} \int_t^T K_6^q(s, t) ds < \infty, \\ M_7 &= \sup_{t \in [0, T]} \int_t^T K_7^2(s, t) ds < \infty, \quad M_8 = \sup_{t \in [0, T]} \int_t^T K_8^q(s, t) ds < \infty. \end{aligned}$$

**Theorem 5.1** *Let (H2) hold, then FBSVIE (27) admits a unique M-solution.*

**Proof.** Let  $\mathcal{M}^2[0, T]$  be the set of element  $(Y, Z)$  in  $\mathcal{H}^2[0, T]$  such that

$$Y(t) = EY(t) + \int_0^t Z(t, s)dW(s), \quad (t, s) \in \Delta.$$

It is easy to see that  $\mathcal{M}^2[0, T]$  is a closed subspace of  $\mathcal{H}^2[0, T]$ , see [27]. We consider the following equation

$$\left\{ \begin{aligned} X(t) &= \varphi(t) + \int_0^t A_1(t, s)x(s)ds + \int_0^t A_2(t, s)x(s)dW(s) \\ &\quad + \int_0^t B_1(t, s) \left( E^{\mathcal{F}_s} \int_s^T [D_1(u, s)y(u)du + D_2(u, s)z(u, s)]du \right) ds \\ &\quad + \int_0^t B_2(t, s) \left( E^{\mathcal{F}_s} \int_s^T [D_1(u, s)y(u)du + D_2(u, s)z(u, s)]du \right) dW(s), \\ Y(t) &= \phi_1(t)x(t) + E^{\mathcal{F}_t} \int_t^T C'_1(s, t)y(s)ds + E^{\mathcal{F}_t} \int_t^T C'_2(s, t)z(s, t)ds, \end{aligned} \right. \quad (28)$$

for any  $\varphi(\cdot) \in L^2_{\mathcal{F}}[0, T]$ , and  $(x(\cdot), y(\cdot), z(\cdot, \cdot)) \in L^2_{\mathcal{F}}[0, T] \times \mathcal{M}^2[0, T]$ . Obviously (28) admits a unique adapted M-solution  $(X(\cdot), Y(\cdot), Z(\cdot, \cdot)) \in L^2_{\mathcal{F}}[0, T] \times \mathcal{M}^2[0, T]$ , and we can define a map  $\Theta : L^2_{\mathcal{F}}[0, T] \times \mathcal{M}^2[0, T] \rightarrow L^2_{\mathcal{F}}[0, T] \times \mathcal{M}^2[0, T]$  by

$$\begin{aligned}\Theta(x(\cdot), y(\cdot), z(\cdot, \cdot)) &= (X(\cdot), Y(\cdot), Z(\cdot, \cdot)), \\ \forall (x(\cdot), y(\cdot), z(\cdot, \cdot)) &\in L^2_{\mathcal{F}}[0, T] \times \mathcal{M}^2[0, T].\end{aligned}$$

Let  $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot, \cdot)) \in L^2_{\mathcal{F}}[0, T] \times \mathcal{M}^2[0, T]$  and

$$\Theta(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot, \cdot)) = (\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, \cdot)).$$

As to the first forward equation in (28),

$$\begin{aligned}& E \int_0^T e^{-\beta t} |X(t) - \bar{X}(t)|^2 dt \\ & \leq 2E \int_0^T e^{-\beta t} \left| \int_0^t A_1(t, s)[x(s) - \bar{x}(s)] + B_1(t, s)[p(s) - \bar{p}(s)] ds \right|^2 dt \\ & \quad + 4E \int_0^T e^{-\beta t} \left( \int_0^t A_2^2(t, s)[x(s) - \bar{x}(s)]^2 ds + \int_0^t B_2^2(t, s)[p(s) - \bar{p}(s)]^2 ds \right) dt \\ & \leq 4(M_1 + M_2)E \int_0^T |x(s) - \bar{x}(s)|^2 \int_s^T e^{-\beta t} dt \\ & \quad + 4(M_3 + M_4)E \int_0^T |p(s) - \bar{p}(s)|^2 e^{2\beta s} \int_s^T e^{-\beta t} dt \\ & \leq \frac{C}{\beta} E \int_0^T |x(s) - \bar{x}(s)|^2 e^{-\beta s} ds + \frac{C}{\beta} E \int_0^T |p(s) - \bar{p}(s)|^2 e^{\beta s} ds,\end{aligned}$$

where we denote

$$p(s) - \bar{p}(s) = E^{\mathcal{F}_s} \int_s^T D_1(u, s)[y(u) - \bar{y}(u)] du + E^{\mathcal{F}_s} \int_s^T D_2(u, s)[z(u, s) - \bar{z}(u, s)] du.$$

Obviously we have

$$\begin{aligned}& E \int_0^T e^{\beta t} |p(t) - \bar{p}(t)|^2 dt \\ & \leq 2M_7 E \int_0^T e^{\beta t} \int_t^T |y(s) - \bar{y}(s)|^2 ds dt + 2M_8 E \int_0^T e^{\beta t} \int_t^T |z(s, t) - \bar{z}(s, t)|^2 ds dt \\ & \leq \frac{2M_7}{\beta} E \int_0^T e^{\beta s} |y(s) - \bar{y}(s)|^2 ds + 2M_8 E \int_0^T e^{\beta s} |y(s) - \bar{y}(s)|^2 ds,\end{aligned}$$

consequently

$$\begin{aligned}& E \int_0^T e^{-\beta t} |X(t) - \bar{X}(t)|^2 dt \\ & \leq \frac{C}{\beta} E \int_0^T |x(s) - \bar{x}(s)|^2 e^{-\beta s} ds + \frac{C}{\beta} E \int_0^T |y(s) - \bar{y}(s)|^2 e^{\beta s} ds,\end{aligned}\tag{29}$$

where  $C$  depends on  $M_i$ ,  $i = 1, 2, 3, 4, 7, 8$ . As to the other one in (28), for some  $p \in (1, 2)$ , and  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$\begin{aligned}
& E \int_0^T e^{\beta t} |Y(t) - \bar{Y}(t)|^2 dt \\
& \leq 2E \int_0^T e^{\beta t} \phi_1^2(t) |x(t) - \bar{x}(t)|^2 dt + C(M_5 + M_7) E \int_0^T e^{\beta t} \int_t^T |y(s) - \bar{y}(s)|^2 ds dt \\
& \quad + C(M_6 + M_8) E \int_0^T e^{\beta t} \left( \int_t^T |z(s, t) - \bar{z}(s, t)|^p ds \right)^{\frac{2}{p}} dt \\
& \leq \frac{1}{2} E \int_0^T e^{-\beta t} |x(t) - \bar{x}(t)|^2 dt + \frac{C}{\beta} E \int_0^T e^{\beta s} |y(s) - \bar{y}(s)|^2 ds \\
& \quad + C \left[ \frac{1}{\beta} \right]^{\frac{2-p}{p}} E \int_0^T dt \int_t^T e^{\beta s} |z(s, t) - \bar{z}(s, t)|^2 ds \\
& \leq C \left( \left[ \frac{1}{\beta} \right]^{\frac{2-p}{p}} + \frac{1}{\beta} \right) E \int_0^T e^{\beta s} |y(s) - \bar{y}(s)|^2 ds + \frac{1}{2} E \int_0^T e^{-\beta t} |x(t) - \bar{x}(t)|^2 dt, \quad (30)
\end{aligned}$$

where  $C$  depends on  $M_i$ ,  $i = 5, 6, 7, 8$ . Note that here we use the following fact, for  $1 < p < 2$ , and  $r > 0$ ,

$$\begin{aligned}
& \left[ \int_t^T |z(s, t) - \bar{z}(s, t)|^p ds \right]^{\frac{2}{p}} \\
& \leq \left[ \int_t^T e^{-rs \frac{2}{2-p}} ds \right]^{\frac{2-p}{p}} \int_t^T e^{rs \frac{2}{p}} |z(s, t) - \bar{z}(s, t)|^2 ds \\
& \leq \left[ \frac{1}{r} \right]^{\frac{2-p}{p}} \left[ \frac{2-p}{2} \right]^{\frac{2-p}{p}} e^{-rt \frac{2}{p}} \int_t^T e^{rs \frac{2}{p}} |z(s, t) - \bar{z}(s, t)|^2 ds.
\end{aligned}$$

By (29) and (30) we obtain

$$\begin{aligned}
& E \int_0^T e^{-\beta t} |X(t) - \bar{X}(t)|^2 dt + E \int_0^T e^{\beta t} |Y(t) - \bar{Y}(t)|^2 dt \\
& \leq \frac{C}{\beta} E \int_0^T |x(s) - \bar{x}(s)|^2 e^{-\beta s} ds + \frac{1}{2} E \int_0^T e^{-\beta t} |x(t) - \bar{x}(t)|^2 dt \\
& \quad + C \left( \left[ \frac{1}{\beta} \right]^{\frac{2-p}{p}} + \frac{1}{\beta} \right) E \int_0^T e^{\beta s} |y(s) - \bar{y}(s)|^2 ds,
\end{aligned}$$

where  $C$  depends on  $M_i$  ( $i = 1, 2, \dots, 8$ ). So we can choose a suitable  $\beta$  such that the mapping  $\Theta$  is contracted and the result holds naturally.  $\square$

Suppose  $G = 0$ ,  $R_{11} > 0$  and  $R_{22} < 0$ , then  $R^{-1}$  exists which is expressed by

$$R^{-1}(t) = \begin{bmatrix} A^{-1}(t) & -A^{-1}(t)R_{12}(t)R_{22}^{-1}(t) \\ -B^{-1}(t)R_{21}(t)R_{11}^{-1}(t) & B^{-1}(t) \end{bmatrix},$$

where  $A(t) = R_{11}(t) - R_{12}(t)R_{22}(t)R_{21}(t)$ ,  $B(t) = R_{22}(t) - R_{21}(t)R_{11}^{-1}(t)R_{12}(t)$ . Moreover we assume  $A^{-1}(t)$ ,  $B^{-1}(t)$ ,  $R_{11}(t)$  and  $R_{22}(t)$  are bounded, thus  $R(t)^{-1}$  is uniformly bounded, then (23) can be rewritten as  $u(t) = -R(t)^{-1}[S(t)X(t) + \lambda(t)]$  with  $t \in [0, T]$ . After substituting  $u(t)$  into (3), (23) and (24), we obtain the following:

$$\left\{ \begin{array}{l} X(t) = \varphi(t) + \int_0^t [(A_1(t, s) - U_1(t, s)R^{-1}(s)S(s))X(s) - U_1(t, s)R^{-1}(s)\lambda(s)]ds \\ \quad + \int_0^t [(A_2(t, s) - U_2(t, s)R^{-1}(s)S(s))X(s) - U_2(t, s)R^{-1}(s)\lambda(s)]dW(s), \\ Y(t) = [Q(t) - S^T(t)R^{-1}(t)S(t)]X(t) - S^T(t)R^{-1}(t)\lambda(t) + \int_t^T A_1(s, t)Y(s)ds \\ \quad + \int_t^T A_2(s, t)Z(s, t)ds - \int_t^T Z(t, s)dW(s), \\ \lambda(t) = E^{\mathcal{F}_t} \int_t^T [U_1^T(s, t)Y(s) + U_2^T(s, t)Z(s, t)] ds, \end{array} \right. \quad (31)$$

where  $U_i(t, s) = (B_i(t, s), C_i(t, s))$  ( $i = 1, 2$ ).

The preceding theorem implies that if  $[Q(t) - S^T(t)R^{-1}(t)S(t)]$  satisfies certain condition, then (31) admits a unique M-solution  $(X, Y, Z, \lambda)$ , thereby the following result is straightforward.

**Theorem 5.2** *Let (H1) hold,  $[Q(t) - S^T(t)R^{-1}(t)S(t)] < \frac{1}{2}e^{-\beta t}$ , where  $\beta$  is a constant depending on the upper boundary of the coefficients in the game problem, moreover,  $R^{-1}(t)$  is bounded, then (31) admits a unique M-solution  $(X, Y, Z, \lambda)$ . Furthermore, if (20) and (21) hold, then the quadratic integral game admits an open-loop saddle point  $\hat{u}$ , and it admits a representation,  $u(t) = -R(t)^{-1}[S(t)X(t) + \lambda(t)]$ .*

## 5.2 Some furthermore considerations on stochastic integral games

In this subsection, we would like to give some furthermore considerations along the routine above. As to the case of  $G \neq 0$ , if we define  $u(t)$  as  $u(t) = -R^{-1}(t)\lambda(t)$ , then (3), (23) and (24) can be rewritten as

$$\left\{ \begin{array}{l} X(t) = \varphi(t) - \int_0^t U_1(t, s)R^{-1}(s)\lambda(s)ds - \int_0^t U_2(t, s)R^{-1}(s)\lambda(s)dW(s) \\ \quad + \int_0^t A_1(t, s)X(s)ds + \int_0^t A_2(t, s)X(s)dW(s), \\ Y(t) = Q(t)X(t) - S(t)^T R^{-1}(t)\lambda(t) + A_1^T(T, t)GX(T) + \int_t^T A_1(s, t)Y(s)ds \\ \quad + A_2^T(T, t)\theta(t) + \int_t^T A_2(s, t)Z(s, t)ds - \int_t^T Z(t, s)dW(s), \\ \lambda(t) = E^{\mathcal{F}_t} [U_1^T(T, t)GX(T) + U_2^T(T, t)\theta(t)] + S(t)X(t) \\ \quad + E^{\mathcal{F}_t} \int_t^T [U_1^T(s, t)Y(s) + U_2^T(s, t)Z(s, t)] ds, \end{array} \right. \quad (32)$$

where  $GX(T) = EGX(T) + \int_0^T \theta(s)dW(s)$ . Obviously (32) is coupled FBSVIE. In some special case, for example,  $S(t) = 0$ ,  $R$  is uniform positive,  $Q$  and  $G$  are non-negative, then the above FBSVIE (32) admits a unique M-solutions, see p.75 in [27]. However, as to the general case, the solvability problem is still a question for us to endeavor to overcome. One main technical obstacle is how to deal with the appearance of  $GX(T)$  in the second equation, nonetheless, it is just the reason, we believe, that the problem has certain relations with the solvability for some stochastic Fredholm-Volterra integral equation. To get some feeling about this, Let us consider a special case below. We assume that all the coefficients aforementioned are deterministic,  $A_i(t, s) = 0$ , ( $i = 1, 2$ ),  $\varphi(\cdot) = \varphi_1(t) + \int_0^t l(t, s)dW(s)$ ,  $\varphi_1$  and  $l$  are deterministic functions. In such special setting,  $Y(t) = Q(t)X(t) - S^T(t)R^{-1}(t)\lambda(t)$ ,  $t \in [0, T]$ , and  $Z(t, s) = 0$ ,  $0 \leq t \leq s \leq T$ . Due to the martingale representation theorem, there must exists a unique process  $\pi$ , such that

$$\lambda(t) = E\lambda(t) + \int_0^t \pi(t, s)dW(s), \quad t \in [0, T], \quad (33)$$

$$X(t) = EX(t) + \int_0^t K(t, s)dW(s), \quad t \in [0, T], \quad (34)$$

thus we can express  $Z(t, s)$ ,  $(t, s) \in \Delta$  by

$$Z(t, s) = Q(t)K(t, s) - S^T(t)R^{-1}(t)\pi(t, s).$$

On the other hand,

$$GX(T) = G\varphi(T) - \int_0^T GU_1(T, s)R^{-1}(s)\lambda(s)ds - \int_0^T GU_2(T, s)R^{-1}(s)\lambda(s)dW(s), \quad (35)$$

then substitute (35) into  $GX(T) = EGX(T) + \int_0^T \theta(s)dW(s)$  and by stochastic Fubini theorem, we have

$$\theta(s) = - \int_s^T GU_1(T, u)R^{-1}(u)\pi(u, s)du - GU_2(T, s)R^{-1}(s)\lambda(s) + l(T, s).$$

Similarly we get

$$K(t, s) = - \int_s^t U_1(t, u)R^{-1}(u)\pi(u, s)du - U_2(t, s)R^{-1}(s)\lambda(s) + l(t, s).$$



Then put the expression of  $X$  and  $Y$  into the third equation of (32) we have

$$\begin{aligned}
\lambda(t) &= \Sigma_1(t) + E^{\mathcal{F}_t} \int_0^T \Sigma_2(t, s) \lambda(s) ds + E^{\mathcal{F}_t} \int_0^T \Sigma_3(t, s) \lambda(s) dW(s) \\
&\quad + E^{\mathcal{F}_t} \int_t^T \Sigma_4(t, s) \pi(s, t) ds + E^{\mathcal{F}_t} \int_t^T \Sigma_5(t, s) \lambda(s) ds \\
&\quad + E^{\mathcal{F}_t} \int_t^T \Sigma_6(t, s) \lambda(s) dW(s) + \Sigma_7(t) \lambda(t) \\
&\quad + E^{\mathcal{F}_t} \int_0^t \Sigma_8(t, s) \lambda(s) dW(s) + E^{\mathcal{F}_t} \int_0^t \Sigma_9(t, s) \lambda(s) ds \\
&= \Sigma_1(t) + E^{\mathcal{F}_t} \int_0^T \Sigma'_2(t, s) \lambda(s) ds + E^{\mathcal{F}_t} \int_0^T \Sigma'_3(t, s) \lambda(s) dW(s) \\
&\quad + E^{\mathcal{F}_t} \int_t^T \Sigma_4(t, s) \pi(s, t) ds + E^{\mathcal{F}_t} \int_t^T \Sigma'_5(t, s) \lambda(s) ds \\
&\quad + E^{\mathcal{F}_t} \int_t^T \Sigma'_6(t, s) \lambda(s) dW(s) + \Sigma_7(t) \lambda(t), \tag{36}
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1(t) &= U_1^T(T, t) G \varphi(T) + \int_t^T U_1^T(s, t) Q(s) \varphi(s) ds + S(t) \varphi(t) \\
&\quad + E^{\mathcal{F}_t} \int_t^T \left[ U_1^T(s, t) Q(s) \int_0^t l(s, u) dW(u) ds + U_2^T(s, t) Q(s) l(s, t) \right] ds \\
\Sigma_2(t, s) &= -G U_1^T(T, t) U_1(T, s) R^{-1}(s); \quad \Sigma_3(t, s) = -G U_1^T(T, t) U_2(T, s) R^{-1}(s); \\
\Sigma_4(t, s) &= -U_2^T(T, t) G U_1(T, s) R^{-1}(s) - U_2^T(s, t) S^T(s) R^{-1}(s) \\
&\quad - \int_s^T U_2^T(u, t) Q(u) U_1(u, s) du R^{-1}(s); \\
\Sigma_5(t, s) &= - \int_s^T U_1^T(u, t) Q(u) U_1(u, s) du R^{-1}(s) - U_1^T(s, t) S(s) R^{-1}(s); \\
\Sigma_6(t, s) &= - \int_s^T U_1^T(u, t) Q(u) U_2(u, s) du R^{-1}(s); \\
\Sigma_7(t, s) &= -U_2^T(T, t) G U_2(T, t) R^{-1}(t) - \int_t^T U_2^T(s, t) Q(s) U_2(s, t) ds R^{-1}(t); \\
\Sigma_8(t, s) &= - \int_t^T U_1^T(u, t) Q(u) U_2(u, s) du R^{-1}(s) - S(t) U_2(t, s) R^{-1}(s); \\
\Sigma_9(t, s) &= - \int_t^T U_1^T(u, t) Q(u) U_1(u, s) du R^{-1}(s) - S(t) U_1(t, s) R^{-1}(s); \\
\Sigma'_2(t, s) &= \Sigma_2(t, s) + \Sigma_9(t, s); \quad \Sigma'_3(t, s) = \Sigma_3(t, s) + \Sigma_8(t, s); \\
\Sigma'_5(t, s) &= \Sigma_5(t, s) - \Sigma_9(t, s); \quad \Sigma'_6(t, s) = \Sigma_6(t, s) - \Sigma_8(t, s).
\end{aligned}$$

We can denote (36) as a linear BSFVIE. To sum up,

**Theorem 5.3** *Let all the coefficients are deterministic,  $A_i(t, s) = 0$ ,  $i = 1, 2$ ,  $\varphi(t) = \varphi_1(t) + \int_0^t l(t, s)dW(s)$ ,  $\varphi_1$  and  $l$  are deterministic functions,  $R^{-1}(t)$  exists and bounded. If (36) admits a solution  $\lambda$ , furthermore, we assume (20) and (21) hold, then the quadratic integral game admits an open-loop saddle point  $\hat{u}$ , and it admits a representation  $\hat{u}(t) = -R^{-1}(t)\lambda(t)$ .*

More specially, suppose that  $U_2 = 0$ , then the preceding BSFVIE (36) becomes

$$\lambda(t) = \Sigma_1(t) + E^{\mathcal{F}_t} \int_0^T \Sigma_2''(t, s)\lambda(s)ds + E^{\mathcal{F}_t} \int_0^t \Sigma_9''(t, s)\lambda(s)ds, \quad (37)$$

where  $\Sigma_2''(t, s) = \Sigma_2(t, s) + \Sigma_5(t, s)$  and  $\Sigma_9''(t, s) = \Sigma_9(t, s) - \Sigma_5(t, s)$ . Equation (37) is a forward stochastic Fredholm-Volterra integral equation (SFVIE for short), thereby under some assumptions the above SFVIE admits a unique solution  $\lambda$ , see [20], [19] and the reference cited therein. Next we will present a example to show the application of the above results.

Let us consider a stochastic delay equation of the form

$$\begin{aligned} dX(t) = & \left[ A_1'(t)X(t) + A_2'(t)X(t-h) + \int_{t-h}^t A_0'(t, s)X(s)ds + B_1'(t)u_1(t) \right. \\ & \left. + B_2'(t)u_1(t-h) + C_1'(t)u_2(t) + C_2'(t)u_2(t-h) \right] + D'(t)dW(t), \end{aligned} \quad (38)$$

with  $t \in [0, T]$  where  $X(t) = k(t)$  with  $t \in [-h, 0]$ ,  $A_j'$ ,  $B_i'$ ,  $C_i'$ ,  $D'$  and  $k$  are bounded deterministic functions, ( $i = 1, 2, j = 0, 1, 2$ ),  $B_2' \equiv 0$ ,  $C_2' \equiv 0$  for  $t < h$ , the delay  $h > 0$ . Notice that when  $A_0' \equiv 0$ ,  $D' \equiv 0$  and we consider the system in a deterministic setting, then (38) will degenerate into the one in Section 7 of [29]. It was shown in [11], see also [16], that this type of delay equation can be written in the following equivalent form:

$$X(t) = X_0(t) + \int_0^t [K_1(t, s)u_1(s) + K_2(t, s)u_2(s)]ds + \int_0^t \Phi(t, s)D'(s)dW(s) \quad (39)$$

where

$$X_0(t) = \Phi(t, 0)k(0) + \int_{-h}^0 \left[ \Phi(t, s+h)A_2'(s+h) + \int_0^h \Phi(t, u)A_0'(u, s)du \right] k(s)ds,$$

$K_1(t, s) = \Phi(t, s)B_1'(s) + \Phi(t, s+h)B_2'(s+h)$ ,  $K_2(t, s) = \Phi(t, s)C_1'(s) + \Phi(t, s+h)C_2'(s+h)$ , and  $\Phi$  is the transition function:

$$\frac{\partial \Phi}{\partial t}(t, s) = A_1'(t)\Phi(t, s) + A_2'(t)\Phi(t-h, s) + \int_{t-h}^t A_0'(t, u)\Phi(u, s)du$$

with  $t \in [0, T]$ ,  $\Phi(s, s) = 1$  and  $\Phi(t, s) = 0$  with  $t < 0$ . Obviously (39) is a simple form of the forward equation in (32). In this case,  $\varphi_1(t) = X_0(t)$ ,  $l(t, s) = \Phi(t, s)D'(s)$ ,  $U_1 = (K_1, K_2)$ , then we get

**Theorem 5.4** *Let the dynamic system is described by a stochastic delay equation (38), and the cost functional is defined by (4),  $R^{-1}$  is bounded. If the SFVIE (37) admits a solution, furthermore, (20) and (21) hold, then the quadratic integral game admits an open-loop saddle point  $\hat{u}$ , and it admits a representation  $\hat{u}(t) = -R^{-1}(t)\lambda(t)$ .*

Furthermore, by assuming  $S(t) = 0$ ,  $R_{12} = R_{21} = 0$ , we can obtain one express for the saddle point by

$$\begin{aligned} u_1(t) &= -R_{11}^{-1}(t) \left[ K_1(T, t)E^{\mathcal{F}_t}GX(T) + E^{\mathcal{F}_t} \int_t^T K_1(s, t)Q(s)X(s)ds \right], \quad t \in [0, T], \\ u_2(t) &= -R_{22}^{-1}(t) \left[ K_2(T, t)E^{\mathcal{F}_t}GX(T) + E^{\mathcal{F}_t} \int_t^T K_2(s, t)Q(s)X(s)ds \right], \quad t \in [0, T]. \end{aligned}$$

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